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Wilson–Polchinski exact renormalization group equation for O(N) systems: leading and next-to-leading orders in the derivative expansion

C Bervillier

Service de Physique Théorique, CEA/DSM/SPhT-CNRS/SPM/URA 2306 CEA/Saclay, F-91191 Gif-sur-Yvette Cédex, France
and
Laboratoire de Mathématiques et Physique Théorique, CNRS/UMR 6083, Université de Tours, Parc de Grandmont, 37200 Tours, France

E-mail: bervil@spht.saclay.cea.fr

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Abstract
With a view to study the convergence properties of the derivative expansion of the exact renormalization group (RG) equation, I explicitly study the leading and next-to-leading orders of this expansion applied to the Wilson–Polchinski equation in the case of the N-vector model with the symmetry O(N). As a test, the critical exponents η and ν as well as the subcritical exponent ω (and higher ones) are estimated in three dimensions for values of N ranging from 1 to 20. I compare the results with the corresponding estimates obtained in preceding studies or treatments of other O(N) exact RG equations at second order. The possibility of varying N allows the derivative expansion method to be better valued. The values obtained from the resummation of high orders of perturbative field theory are used as standards to illustrate the eventual convergence in each case. Particular attention is drawn to the preservation (or not) of the reparameterization invariance.

1. Introduction

The renormalization group (RG) theory is suitable for the study of many modern physical problems. Generically, every situation where the scale of typical physical interest belongs to a (wide) range of correlated or coupled scales may be (must be?) treated by RG techniques. Critical phenomena, which are characterized by one (or several) diverging correlation length(s), provide 'the' didactic example [1]. Quantum field theory, with its strongly correlated quantum fluctuations, is not less famous since it has given rise to the early stages of the RG theory [2].

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Thanks to a fortunate success (essentially due to an impressive diagrammatic calculation [3]) in estimating the critical behaviour of some systems [4, 5], the perturbative framework has pushed into the background the undoubtedly nonperturbative character [6–8] of the RG theory. As a consequence, there has been relatively little interest in the development of nonperturbative RG techniques [9]. In particular the formulation of the RG theory via an infinitesimal change of the scale of reference (running scale), designated by the generic expression 'exact renormalization group equation' [1] although known since 1971 [10], has actually been actively considered only since the beginning of the 1990s [9]. Because the variety of systems to which the exact RG formulation could be applied is large [9, 11] (see also [12]) and also because the perturbative framework is generally not well adapted to such studies [8, 13, 14], it is worthwhile making every endeavour to master, if possible, the exact RG framework.

The exact RG equation is an integro-differential equation, the study of which calls for approximations and/or truncations. Among the possible approximations, those based on expansions in powers of a small parameter such as $\epsilon = d_{\mathrm{u}} - d$ or $1/N$ (where the upper-dimension $d_{\mathrm{u}} = 4$ for the N-vector model) are perturbative in essence. They, however, present the advantage of allowing analytic calculations but are attached to the smallness of quantities that are actually not small in the cases of physical interest. In some cases the perturbative framework may fail [14, 8].

The derivative expansion [15], of present interest here, is an expansion in powers of the derivative of the field. It is not associated to a small parameter, though it is expected to be rather adapted to the study of phenomena at small momenta[1] (large distances) like critical phenomena, for instance. The interest of the derivative expansion is that the physical parameters (like d and N) may take on arbitrary values. Hence, in the range of validity of the expansion (thus presumed to be in the large distance regime), the approach is actually nonperturbative. The drawback is the necessary recourse to numerical techniques (for studying coupled nonlinear ordinary differential equations (ODEs)) that are not always well controlled. Consequently, very few orders of the derivative expansion have really been explicitly considered.

Many studies have been effectuated in the local potential approximation (LPA), i.e., at the leading order $[\mathrm{O}(\partial^0)]$ of the derivative expansion [9, 11, 16]. Also, estimates of the critical exponents for Ising-like models ($N = 1$) have been obtained several times from full studies of the next-to-leading order[2] (i.e., $\mathrm{O}(\partial^2)$) [15, 21–24, 13], and even from a full study of the third order $[\mathrm{O}(\partial^4)]$ [25]. In contrast, only two full studies have been effectuated up to $\mathrm{O}(\partial^2)$ for the $\mathrm{O}(N)$ vector model [26, 27]. Yet, this model provides us with the opportunity of varying N and of comparing the results with the best estimates of critical exponents obtained from six [28, 29] or seven [30] orders of the perturbative framework in a wide range of values of N. Interesting information on the possible convergence of the derivative expansion is then reachable.

In the present study, I consider the $\mathrm{O}(N)$ Wilson–Polchinski exact RG equation expanded up to order $\mathrm{O}(\partial^2)$ in the derivative expansion, and I calculate the critical exponents. Beyond the specific calculation (which was lacking) the real aim is to try to clarify (and also to evaluate) the present status of the derivative expansion. Still, I must explain why I consider the Wilson–Polchinski equation.

[1] A recent interesting attempt to adapt the derivative expansion to phenomena that include effects at larger momenta is made in [17].

[2] In incomplete studies some contributions to a given order of the derivative expansion are neglected. For example, in [18], despite an estimation of the critical exponent η (which is exactly equal to zero at order ∂^0), the order ∂^2 has not been completely considered because the evolution equation of the wavefunction renormalization has been neglected. In [19], $\mathrm{O}(N)$ systems are incompletely studied up to $\mathrm{O}(\partial^2)$ because one differential equation has been discarded. Also, the study of [20] at order ∂^4, though interesting, is incomplete since only three differential equations, among five constituting actually the order ∂^4, have been treated.

Indeed, there are several different approaches and treatments of the exact RG equation, so it is not easy to really estimate the validity of the choice of the equation or/and of the calculations available in the literature. Let me try to briefly summarize the situation and to justify my choice.

There are two families of exact RG equations (for a review see [9]). The first family expresses the RG flow ($\Lambda\, \mathrm{d}S_\Lambda/\mathrm{d}\Lambda$) of the microscopic action $S_\Lambda[\phi]$ (Hamiltonian) associated to a running momentum scale Λ which, in the circumstances, is a running ultra-violet cutoff. The second family expresses the RG flow of $\Gamma_\Lambda[M]$, the Legendre transform of $S_\Lambda[\phi]$; in that case the running scale Λ effectively appears as an infra-red momentum cutoff.

There is no fundamental difference between the two families since the object of the RG is the same in the two cases: accounting for all the correlated scales Λ ranging from 0 to ∞ (at criticality); only the physical meaning of the field variable at hand has been changed: ϕ is related to a microscopic description (like a spin of the Ising model) while M is thought of as a macroscopic variable (like the magnetization). If one wants to calculate an equation of state or some correlation functions or some universal critical amplitude ratios, the second family of equations is better adapted. But if one only wants to estimate critical exponents (for example to illustrate the convergence of the derivative expansion), then considering the first family is surely more efficient. Actually, the set of ODEs generated in the derivative expansion is much simpler when considered with the first family than with the second[3]. The first family is indeed better adapted to the calculation of critical exponents for the same reason as in the field theoretical approach to critical phenomena [4]: the critical exponents are defined from renormalization functions that are introduced within the microscopic action $S_\Lambda[\phi]$. The Wilson [1] and Polchinski [33] exact RG equations belong to the first family; they only differ by the way the smooth cutoff function has been defined (a specific function for Wilson, an arbitrary one for Polchinski). Of course, the two equations are physically equivalent, but due to a misunderstanding in the introduction of the critical exponent η in the exact RG equation as formulated by Polchinski, it is only recently that the equivalence has been clearly established [34, 13].

If the coupled set of ODEs generated by the derivative expansion is different in the two families of exact RG equations, the treatments of the differential equations encountered in the literature differ also. For convenience, let me classify the studies in two groups according to whether the authors have adopted the conventional approach (defined below) or not.

The conventional approach is characterized as follows:

(i) the set of ODEs is numerically studied as such (e.g., without considering any artefact such as an expansion in powers of the field).
(ii) The critical exponent η is introduced in a conventional way as defined in [1, 35, 13].
(iii) The critical exponents are estimated from a set of eigenvalue equations linearized about a fixed point solution of the flow equation.
(iv) The reparameterization invariance is explicitly accounted for.

To limit myself to the studies mentioned above that consider $\mathrm{O}(N)$ systems via equations of the second family developed up to $\mathrm{O}(\partial^2)$ [26, 27]: the study of [26] follows the conventional approach while that of [27] does not. In fact, in this latter work, except the first point above, none of the other points is satisfied. This is particularly important relatively to point (iv)

[3] In the study of [25] of the Ising case up to $\mathrm{O}(\partial^4)$, the consideration of an exact RG equation of the second family yields a set of ODEs the writing of which requires 20 pages [31] while, even at order ∂^6, the first family yields a set of equations that holds on a half of page [32]. The complexity of dealing with the second family is also well illustrated in the appendix of [26].

because the reparameterization invariance induces a line of equivalent fixed points along which η is constant [36]. In the case where the invariance is broken (it is generally the case within the derivative expansion, except in [26]) then the fixed points along the line are no longer equivalents, and the effective η (when introduced conventionally) varies with the global normalization of the field ϕ. Nevertheless, even if the invariance is broken, one expects that a vestige of this invariance can still be observed [36, 37, 15] via an extremum of η on varying the global normalization of the field ϕ. The absence of explicit consideration of the reparameterization invariance in [27] is intriguing in as much as the estimation of the critical exponents is excellent (see section 3.3).

By considering the Wilson–Polchinski RG equation for O(N) systems, my first aim is to illustrate the conventional treatment as described above. Additional aims follow:

(i) Morris and Turner [26] have imposed the reparameterization invariance by choosing a specific cutoff function and the resulting estimates of the critical exponents are not very good, especially for ω. Does the Wilson–Polchinski RG equation also produce such bad results at order O(∂^2)?
(ii) Is the Wilson–Polchinski RG equation able to produce estimates of critical exponents comparable to those obtained by Gersdorff and Wetterich [27]?
(iii) The O(N) exact RG equation expanded up to order ∂^2 involves three coupled ODEs. Compared to the Ising case $N = 1$, it is a somewhat intermediate state between the order ∂^2 which involves two coupled equations and the order ∂^4 which involves five coupled equations [25, 20].
(iv) Does one observe some signs of convergence of the derivative expansion already at order ∂^2?

Relative to this latter point, it is worth mentioning here the work of Litim [38], whose aim, though very interesting, differs from that of the present work. Litim focuses its attention almost exclusively on the second family of the exact RG equations and especially on the arbitrariness introduced by the regularization process (cutoff procedure) but he does not account for the reparameterization invariance. From general arguments (independent from the derivative expansion), he provides us with a criterion for choosing the regulator which should optimize the convergence of the derivative expansion already at a very low order. The considerations are surely useful especially for studying sophisticated systems, such as gauge field theory for example, for which already the leading order of the derivative expansion is difficult to implement. However, Litim has not actually studied the convergence properties of the derivative expansion in itself but, in fact, has implicitly assumed that it converges (at least, the fact that the expansion could yield only asymptotic series has been excluded). Furthermore, Litim's criterion of choice does not apply to the Wilson–Polchinski RG equation. In particular, at leading order of the derivative expansion, his optimization provides the same estimates of critical exponents as those obtained with the Wilson–Polchinski RG equation (see section 3.2) which, at this order, does not display any dependence on the regularization process.

2. Derivative expansion up to O(∂^2)

2.1. Flow equations

According to [13], after subtraction of the high temperature fixed point $\frac{1}{2}\sum_{a=1}^{N}\int_q (\tilde{P}\psi^2)^{-1}\phi^a_q\phi^a_{-q}$ from the action, the Wilson–Polchinski exact RG equation satisfied by an

O(N)-symmetric action $S[\phi^a]$ (with $a = 1, \ldots, N$) reads as follows:

$$\dot{S} = \sum_{a=1}^{N} \left\{ -\int_q \phi_q^a \left(\tilde{d}_\phi + 2q^2 \frac{\psi'}{\psi} + \mathbf{q} \cdot \partial_q \right) \frac{\delta S}{\delta \phi_q^a} + \int_q (\varpi \tilde{P} - q^2 \tilde{P}') \psi^2 \left[\frac{\delta^2 S}{\delta \phi_q^a \delta \phi_{-q}^a} - \frac{\delta S}{\delta \phi_q^a} \frac{\delta S}{\delta \phi_{-q}^a} \right] \right\} \tag{1}$$

in which $\dot{S}$ stands for $\mathrm{d}S/\mathrm{d}t = -\Lambda\, \mathrm{d}S/\mathrm{d}\Lambda$ (hence $\exp(-t) = \Lambda/\Lambda_0$ in which Λ_0 is some initial momentum scale of [13]), $\mathbf{q}$ is a dimensionless d-vector (d is the dimension of the Euclidean space and $\mathbf{q} = \{q_i, i = 1, \ldots, d\}$), $q^2 = \sum_{i=1}^{d} q_i^2$, $\tilde{P}(q^2)$ is a dimensionless cutoff function that decreases rapidly when $q \to \infty$ with $\tilde{P}(0) = 1$, $\psi(q^2)$ is an arbitrary function (except the normalization $\psi(0) = 1$) introduced to test the reparameterization invariance, $\varpi = 1 - \eta/2$ and $\tilde{d}_\phi = d/2 + \varpi$. A prime denotes a derivative with respect to q^2: $\psi' = \mathrm{d}\psi/\mathrm{d}q^2$, $\tilde{P}' = \mathrm{d}\tilde{P}/\mathrm{d}q^2$ and $\mathbf{q} \cdot \partial_q f(q) = \sum_{i=1}^{d} q_i \partial f / \partial q_i$.

The expansion up to order ∂^2 consists in projecting equation (1) onto actions of the form:

$$S[\phi] = \int \mathrm{d}^d x\, [V(\rho, t) + Z(\rho, t)(\partial \vec{\phi})^2 + Y(\rho, t)(\vec{\phi} \partial \vec{\phi})^2]$$

with

$$\rho = \tfrac{1}{2} \sum_{\alpha=1}^{N} \phi^\alpha \phi^\alpha = \tfrac{1}{2} \vec{\phi}^2.$$

Then the flow equations for V, Z and Y read as follows:

$$\begin{aligned}
\dot{V} &= I_0 \left(NV' + 2\rho V'' \right) + dV - (d + 2\varpi)\, \rho V' - 2\varpi \rho V'^2 + 2I_1 \, (NZ + 2\rho Y) \\
\dot{Z} &= I_0 \left(NZ' + 2\rho Z'' + 2Y \right) - 2\,(\varpi + 1)\, Z - (d + 2\varpi)\, \rho Z' - 2\psi_0' V' \\
&\quad - 4\varpi \left(V'Z + \rho V'Z' \right) - \left[(\varpi - 1)\, \tilde{P}_0' + 2\varpi\, \psi_0' \right] V'^2 \\
\dot{Y} &= I_0 \left(NY' + 2\rho Y'' + 4Y' \right) - (d + 2 + 4\varpi) Y - (d + 2\varpi)\, \rho Y' - 2\psi_0' V'' \\
&\quad - 4\varpi \left(V''Z + 2V'Y + \rho V'Y' + 2\rho V''Y \right) \\
&\quad - 2 \left[(\varpi - 1)\, \tilde{P}_0' + 2\varpi\, \psi_0' \right] \left(V''V' + \rho V''^2 \right)
\end{aligned} \tag{2}$$

in which a prime acting on V, Z or Y denotes this time a derivative with respect to ρ, while $\psi_0' \equiv \psi'(0)$, $\tilde{P}_0' \equiv \tilde{P}'(0)$ and:

$$I_0 = \int_q \left(\varpi \tilde{P} - q^2 \tilde{P}' \right) \psi^2, \qquad I_1 = \int_q q^2 \left(\varpi \tilde{P} - q^2 \tilde{P}' \right) \psi^2. \tag{3}$$

It is convenient to perform the following changes:

$$\rho = N I_0 \bar{\rho}, \qquad V = N I_0\, \bar{V}, \qquad Z = \bar{Z}, \qquad Y = \frac{1}{N I_0} \bar{Y} \tag{4}$$

and then to consider the new set of functions:

$$\mathrm{v}_1 = \frac{\mathrm{d}\bar{V}}{\mathrm{d}\bar{\rho}}, \qquad \mathrm{v}_2 = \bar{Z}, \qquad \mathrm{v}_3 = \bar{Y}.$$

Using these new notations and restoring the writing $\bar{\rho} \longrightarrow \rho$, the set of equations (2) becomes:

$$\dot{\mathrm{v}}_1 = \left(1 + \frac{2}{N} \right) \mathrm{v}_1' + \frac{2}{N} \rho \mathrm{v}_1'' - e \mathrm{v}_1 - (d + e)\, \rho \mathrm{v}_1' - e \left(\mathrm{v}_1^2 + 2\rho \mathrm{v}_1 \mathrm{v}_1' \right) + P_1 \left[\mathrm{v}_2' + \frac{2}{N} \left(\mathrm{v}_3 + \rho \mathrm{v}_3' \right) \right] \tag{5}$$

$$\dot{\mathrm{v}}_2 = \mathrm{v}_2' + \frac{2}{N}\rho \mathrm{v}_2'' + \frac{2}{N}\mathrm{v}_3 - (e+2)\,\mathrm{v}_2 - (d+e)\,\rho \mathrm{v}_2' + 2u\mathrm{v}_1 - 2e\left(\mathrm{v}_1\mathrm{v}_2 + \rho \mathrm{v}_1\mathrm{v}_2'\right) + P_2\mathrm{v}_1^2 \tag{6}$$

$$\dot{\mathrm{v}}_3 = \left(1+\frac{4}{N}\right)\mathrm{v}_3' + \frac{2}{N}\rho \mathrm{v}_3'' - (d+2+2e)\mathrm{v}_3 - (d+e)\,\rho \mathrm{v}_3' + 2u\mathrm{v}_1' - 2e\left[\mathrm{v}_1'\left(\mathrm{v}_2 + 2\rho \mathrm{v}_3\right) + \mathrm{v}_1\left(2\mathrm{v}_3 + \rho \mathrm{v}_3'\right)\right] + 2P_2\left(\mathrm{v}_1 + \rho \mathrm{v}_1'\right)\mathrm{v}_1' \tag{7}$$

in which:

$$u = -\psi_0', \qquad e = 2\varpi$$

$$P_1 = 2\frac{I_1}{I_0}, \qquad P_2 = -\left(\frac{e}{2}-1\right)\tilde{P}'(0) + eu.$$

2.2. *Fixed point equations*

The fixed point equations correspond to the three simultaneous conditions $\dot{\mathrm{v}}_i = 0$ for $i = 1, 2, 3$ which yield three coupled nonlinear ODEs of second order each:

$$\mathrm{v}_1'' = \frac{N}{2\rho}\left[e\mathrm{v}_1\left(1+\mathrm{v}_1\right) - \left(1+\frac{2}{N}\right)\mathrm{v}_1' - P_1\left(\mathrm{v}_2' + \frac{2}{N}\mathrm{v}_3\right)\right] + \frac{N}{2}\left(d+e+2e\mathrm{v}_1\right)\mathrm{v}_1' - P_1\mathrm{v}_3' \tag{8}$$

$$\mathrm{v}_2'' = \frac{N}{2\rho}\left[\left(e+2+2e\mathrm{v}_1\right)\mathrm{v}_2 - \mathrm{v}_2' - \frac{2}{N}\mathrm{v}_3 - 2u\mathrm{v}_1 - P_2\mathrm{v}_1^2\right] + \frac{N}{2}\left[(d+e) + 2e\mathrm{v}_1\right]\mathrm{v}_2' \tag{9}$$

$$\mathrm{v}_3'' = \frac{N}{2\rho}\left[\left(d+2+2e+4e\mathrm{v}_1\right)\mathrm{v}_3 - \left(1+\frac{4}{N}\right)\mathrm{v}_3' + 2\left(e\mathrm{v}_2 - u - P_2\mathrm{v}_1\right)\mathrm{v}_1'\right] + \frac{N}{2}\left[\left(d+e+2e\mathrm{v}_1\right)\mathrm{v}_3' + 2\left(2e\mathrm{v}_3 - P_2\mathrm{v}_1'\right)\mathrm{v}_1'\right]. \tag{10}$$

The differential system is of order six, thus the general solution depends on six arbitrary constants. Three of these constants are fixed so as to avoid the singularity at the origin $\rho = 0$ displayed by the equations, hence the three following conditions:

$$\mathrm{v}_2'(0) = \left[e+2+2e\mathrm{v}_1(0)\right]\mathrm{v}_2(0) - \frac{2}{N}\mathrm{v}_3(0) - \mathrm{v}_1(0)\left[2u + P_2\mathrm{v}_1(0)\right] \tag{11}$$

$$\mathrm{v}_1'(0) = \frac{N}{N+2}\left\{e\mathrm{v}_1(0)\left[1+\mathrm{v}_1(0)\right] - P_1\left[\mathrm{v}_2'(0) + \frac{2}{N}\mathrm{v}_3(0)\right]\right\} \tag{12}$$

$$\mathrm{v}_3'(0) = \frac{N}{N+4}\{[d+2+2e+4e\mathrm{v}_1(0)]\mathrm{v}_3(0) + 2[e\mathrm{v}_2(0) - u - P_2\mathrm{v}_1(0)]\mathrm{v}_1'(0)\}. \tag{13}$$

If η is *a priori* fixed, then the general solution of the set of equations (8)–(10) depends on the three remaining arbitrary constants: for example, the values $\mathrm{v}_i(0)$ for $i = 1, 2, 3$. In general the corresponding solutions are singular at some varying ρ^* (moving singularity), with:

$$\mathrm{v}_1(\rho) \propto \left(\rho^* - \rho\right)^{-1}, \qquad \mathrm{v}_2(\rho) \propto \left(\rho^* - \rho\right)^{-2}, \qquad \mathrm{v}_3(\rho) \propto \left(\rho^* - \rho\right)^{-2}.$$

However, the equations (8)–(10) admit another kind of solution that goes to infinity ($\rho \to \infty$) without encountering any singularity and which behaves asymptotically for large ρ as follows:

$$\mathrm{v}_{1\mathrm{asy}}(\rho) = G_1\rho^{\theta_1} + \left(1+2\theta_1\right)G_1^2\rho^{2\theta_1} + \cdots \tag{14}$$

$$\mathrm{v}_{2\mathrm{asy}}(\rho) = uG_1\rho^{\theta_1} + G_1^2\left(1+\theta_1\right)\frac{2\,(d-e)\,u + (d+e)\,\tilde{P}_0'}{2\,d}\rho^{2\theta_1} + G_2\rho^{\theta_2} + \cdots \tag{15}$$

$$\mathrm{v}_{3\mathrm{asy}}\,(\rho) = u\theta_1 G_1 \rho^{\theta_1 - 1} + 2G_1^2\,\theta_1 \frac{2\,(d-e)\,u + d\,\tilde{P}'_0}{2\,(d+e)} \rho^{2\theta_1 - 1} + G_3 \rho^{\theta_3} + \cdots \tag{16}$$

with:

$$\theta_1 = -\frac{e}{d+e}, \qquad \theta_2 = -\frac{e+2}{d+e}, \qquad \theta_3 = \theta_2 - 1.$$

The values of the three constants $\{G_i; i = 1, 2, 3\}$, correspond to some adjustment of the set $\{\mathrm{v}_i(0); i = 1, 2, 3\}$ and vice versa. This nonsingular solution is the fixed point solution which we are interested in. When η is *a priori* fixed, the six arbitrary constants of integration are then determined, and the differential system is balanced.

If η is considered as an unknown parameter to be determined, then one of the three preceding quantities $\{\mathrm{v}_i(0)\}$ or $\{G_i\}$ must be promoted to the rank of a fixed parameter chosen *a priori*. In general one chooses

$$\mathrm{v}_2(0) = Z_0 \tag{17}$$

which corresponds to having fixed to Z_0 the value of the kinetic term in $S[\phi^a]$ and is customarily associated with the arbitrary global normalization of the field ϕ. One thus obtains a function $\eta(Z_0)$ which should be a constant if the reparameterization invariance of the exact RG equation was preserved by the derivative expansion presently considered (η should be a constant along a line of equivalent fixed points generated by the variation of Z_0). Since it is not the case, one actually obtains a nontrivial function $\eta(Z_0)$. Fortunately a vestige of the reparameterization invariance is preserved and $\eta(Z_0)$ displays an extremum in Z_0. This provides us with an optimal value (η^{opt}) of η (and similarly for Z_0) via the condition:

$$\frac{\mathrm{d}\eta^{\mathrm{opt}}}{\mathrm{d}Z_0} = 0. \tag{18}$$

Instead of using this condition to determine η^{opt}, I use the fact that the line of equivalent fixed points is associated with a redundant operator with a zero eigenvalue[4] [36, 37]. Hence, one may determine η^{opt} by imposing that the fixed point of interest be associated to a zero eigenvalue. This leads us to the consideration of the system of eigenvalue equations.

2.3. Eigenvalue equations

The eigenvalue equations are obtained by linearization of the flow equations (5)–(7) about a fixed point solution $\{\mathrm{v}_i^*, i = 1, 2, 3\}$:

$$\mathrm{v}_i = \mathrm{v}_i^* + \varepsilon \mathrm{e}^{\lambda t} \mathrm{g}_i.$$

Keeping the linear contribution in ε, the following set of coupled ODEs comes:

$$\mathrm{g}_1'' = \frac{N}{2\rho}\left[\left(\lambda + e + 2e\mathrm{v}_1^*\right)\mathrm{g}_1 - \left(1 + \frac{2}{N}\right)\mathrm{g}_1' - P_1\left(\mathrm{g}_2' + \frac{2}{N}\mathrm{g}_3\right)\right] + \frac{N}{2}\left[\left(d + e + 2e\mathrm{v}_1^*\right)\mathrm{g}_1' + 2e\mathrm{g}_1\mathrm{v}_1^{*\prime}\right] - P_1\mathrm{g}_3' \tag{19}$$

[4] The fact that the eigenvalue takes on a definite value although it is associated with a redundant operator is not in conflict with the work of Wegner (see [39] and references therein) which indicates that the eigenvalue of a redundant operator generally varies with the renormalization process. As shown in [36], the linear character of the renormalization of the field in the process of generating the exact RG equation implies a definite eigenvalue (the reparameterization invariance is a direct consequence of this linearity). In contrast, in the case of a nonlinear renormalization scheme, the eigenvalue in question no longer is constant, but depends on the renormalization procedure [36] in accordance with [39].

$$g_2'' = \frac{N}{2\rho}\left[\left(\lambda + e + 2 + 2e\mathrm{v}_1^*\right) g_2 - g_2' - \frac{2}{N} g_3 + 2\left(e\mathrm{v}_2^* - u - P_2\mathrm{v}_1^*\right) g_1\right] + \frac{N}{2}\left[\left(d + e + 2e\mathrm{v}_1^*\right) g_2' + 2eg_1\mathrm{v}_2^{*\prime}\right] \tag{20}$$

$$g_3'' = \frac{N}{2\rho}\Big[(\lambda + d + 2 + 2e + 4e\mathrm{v}_1^*)g_3 - \left(1 + \frac{4}{N}\right)g_3' - 2ug_1' + 2e(g_1'\mathrm{v}_2^* + 2g_1\mathrm{v}_3^*) - 2P_2(g_1\mathrm{v}_1^{*\prime} + \mathrm{v}_1^* g_1') + 2e\mathrm{v}_1^{*\prime} g_2\Big] + \frac{N}{2}[(d + e + 2e\mathrm{v}_1^*)g_3' + 2e(2g_1'\mathrm{v}_3^* + g_1\mathrm{v}_3^{*\prime} + 2\mathrm{v}_1^{*\prime} g_3) - 4P_2 g_1'\mathrm{v}_1^{*\prime}]. \tag{21}$$

Similar considerations to those relative to the determination of the six integration constants associated to the fixed point equations (8)–(10) stand. For a given fixed point solution (v_i^*), there remain six constants to be determined. Three of them are fixed so as to avoid the singularity at the origin $\rho = 0$ displayed by the equations:

$$g_2'(0) = [\lambda + e + 2 + 2e\mathrm{v}_1^*(0)]g_2(0) - \frac{2}{N} g_3(0) + 2[e\mathrm{v}_2^*(0) - u - P_2\mathrm{v}_1^*(0)]g_1(0)$$

$$g_1'(0) = \frac{N}{N+2}\left\{\left[\lambda + e + 2e\mathrm{v}_1^*(0)\right] g_1(0) - P_1\left[g_2'(0) + \frac{2}{N} g_3(0)\right]\right\}$$

$$g_3'(0) = \frac{N}{N+4}\{[\lambda + d + 2 + 2e + 4e\mathrm{v}_1^*(0)]g_3(0) - 2ug_1'(0) + 2e[g_1'(0)\mathrm{v}_2^*(0) + 2g_1(0)\mathrm{v}_3^*(0)] - 2P_2[g_1(0)\mathrm{v}_1^{*\prime}(0) + \mathrm{v}_1^*(0)g_1'(0)] + 2e\mathrm{v}_1^{*\prime}(0)g_2(0)\}.$$

One is interested in the solution that is regular when $\rho \to \infty$.

For λ *a priori* fixed, the three values $\{g_i(0), i = 1, 2, 3\}$ at the origin $\rho = 0$ must be adjusted so that the solution reaches the following regular asymptotic behaviour:

$$\begin{aligned} g_{1\mathrm{asy}} &= S_1\rho^{\chi_1} + 2\left(1 + \theta_1 + \chi_1\right) G_1\, S_1\rho^{\theta_1 + \chi_1} + \cdots \\ g_{2\mathrm{asy}} &= S_2\rho^{\chi_2} + uS_1\rho^{\chi_1} + \cdots \\ g_{3\mathrm{asy}} &= S_3\rho^{\chi_3} + uS_1\chi_1\rho^{\chi_1 - 1} + \cdots \end{aligned}$$

with:

$$\chi_1 = -\frac{\lambda + e}{d + e}, \qquad \chi_2 = -\frac{\lambda + e + 2}{d + e}, \qquad \chi_3 = \chi_2 - 1$$

and the value of the set of constants $\{S_i, i = 1, 2, 3\}$ entering the regular solution at large ρ corresponds to the value of $\{g_i(0)\}$ adjusted at the origin and vice versa.

As in any eigenvalue problem, the global normalization of the eigenvector may be chosen at will so that, fixing $g_1(0) = 1$ for instance, allows one to determine discrete values of λ. Positive values give the critical exponents, while negative values are subcritical (or correction-to-scaling) exponents.

The peculiar value $\lambda = 0$, if present, is associated to the vestige of the reparameterization invariance [37, 15]. Indeed this zero eigenvalue is associated to the redundant operator that generates the line of equivalent fixed points in the complete theory [36, 37]. Conversely, if one considers together the fixed point equations (8)–(10) with the eigenvalue equations (19)–(21) in which λ is set equal to zero (and the condition $g_1(0) = 1$ is maintained), then the condition (17) may be abandoned and $\mathrm{v}_2(0)$ adjusted so as to get a common solution to the set of six coupled ODEs. Then, the resulting value of η necessarily coincides with η^{opt} as defined by equation (18) and the resulting value of $\mathrm{v}_2(0)$ gives Z_0^{opt}. Though the number of differential equations has increased twofold this procedure of determining the optimized fixed point is the most efficient one when parameters (like N and some other ones; see the following section) have to be varied.

2.4. The free parameters

In order to perform an actual numerical study of the set of second order ODEs described in the preceding section, I make the following choice[5]:

$$\tilde{P}(q^2) = \mathrm{e}^{-q^2} \tag{22}$$

$$\psi(q^2) = \frac{1}{1+bq^2}. \tag{23}$$

Following the terminology of [39], the free parameter b is redundant and is intended to be used to optimize the numerical results of the derivative expansion.

The introduction of b is linked to the general property of reparameterization invariance which is broken by the present derivative expansion. The normalization $\psi(0) = 1$ is chosen in order to distinguish the effect of simply changing the global normalization of the field which induces a line of equivalent fixed points (at fixed b). That line is customarily associated to the arbitrariness of Z_0, the value of $Z(0)$ ($\equiv \mathrm{v}_2(0)$), i.e., the coefficient of the kinetic term in $S[\phi^a]$ (see the preceding section). Changing the value of b in the complete theory would induce new (equivalent) lines of equivalent fixed points.

Though they are part of the same invariance, the two free parameters b and Z_0 have effects of different nature in the derivative expansion. Since it is a global constant of normalization, one can expect that, at a given order of the derivative expansion, the variation of Z_0 will still reveal a vestige of the invariance of the exact theory (see the preceding section). In contrast, the effect of b spreads out over different orders of the derivative expansion. Consequently, one expects to observe a progressive restoration of the redundant character of b as the order of the expansion increases. Regarding the extremely low order considered here ($\mathrm{O}(\partial^2)$), one must not expect too much from varying b (see section 3.2).

Notice that the cutoff function $\tilde{P}(q^2)$ is essentially a regulator of the integrals (3) generated by the derivative expansion. Besides the arbitrary choice in the decreasing at large q, the other sources of arbitrariness of the cutoff function may be included within the arbitrary function $\psi(q^2)$. This is why the choice (22) does not involve any free parameter.

3. Numerical study and results

There are two different methods for numerically studying systems of coupled nonlinear ODEs such as those described above: the shooting and the relaxation methods (see, for example [40]). Because it is the easiest to implement, only the shooting method is considered here (though it is less numerically stable than the relaxation method).

3.1. The shooting method

Considering an initial point ρ_1 where known conditions (initial conditions) are imposed and trial values are given to the remaining integration constants, one integrates the ODE system up to a second point ρ_2 where the required conditions are checked. Using a Newton–Raphson algorithm, one iterates the test until the latter conditions are satisfied within a given accuracy.

[5] In [13], I considered one additional parameter within the cutoff function $\tilde{P}$ (named a); however, the equations were invariant in the change $b \to b/a$ so that one of the two parameters was unnecessary.

In the present study, the two points ρ_1 and ρ_2 are either the origin $\rho = 0$ and a large value $\rho_{\rm asy}$ (shooting from the origin) or the reverse (shooting to the origin).

In principle, shooting to the origin is technically better adapted to the present study. For example, in the case of the fixed point equations (8)–(10) with η fixed, one starts from $\rho_{\rm asy}$ with trial values for the three constants $\{G_i\}$ and the initial values of the three functions and of their first derivatives $\{{\rm v}_i, {\rm v}_i'\}$ defined by (14)–(16). After integration up to the origin, one checks whether the three conditions (11)–(13) are fulfilled or not. The system is balanced and the values of the other interesting parameters are simply by-products of the adjustment. For example, the value of Z_0 associated to the arbitrarily fixed value of η is simply read as the value that ${\rm v}_2(0)$ takes on after achievement of the adjustment. If one wants instead to determine $\eta(Z_0)$, then η has to be considered as a trial parameter like $\{G_i\}$, and the supplementary condition (17), with Z_0 an arbitrary fixed number, must be fulfilled at the origin.

Unfortunately, equations (8)–(10) are singular at $\rho = 0$, so it is impossible to shoot to the origin. In [41] (where the leading order LPA was studied for small values of N), the difficulty was circumvented by shooting to a point close to the origin. In contrast, I have chosen to shoot from the origin because it is easy to control the equations starting from that point.

Starting from the origin implies that, for a given value of η, the adjustable parameters no longer are the $\{G_i\}$ but the initial values $\{{\rm v}_i(0)\}$; the initial values of the first derivatives follow from (11)–(13). At the point $\rho_{\rm asy}$ only three conditions are needed in order to balance the number of adjustable parameters. Thus one must eliminate the $\{G_i\}$ from the six equations (14)–(16) to obtain three anonymous conditions. Consequently, the precise knowledge of the asymptotic behaviour of the regular fixed point solution is not necessary.

It appears that imposing conditions like $\{{\rm v}_i''(\rho_{\rm asy}) = 0\}$ is sufficient to determine the solution of interest. I first adjust the trial parameters so as to reach some not too large a value of $\rho_{\rm asy}$, then I increase $\rho_{\rm asy}$ until the desired accuracy is reached on the trial parameters.

3.2. Numerical results

At the leading order LPA, the function ${\rm v}_1$ is considered alone. Only one second order ODE defines the fixed point (twice this number for the eigenvalue problem). The study has already been done in [41] for $d = 3$ and $N = 1$–4. I have extended the results (for $d = 3$) to larger values of N. The study is simple because:

(i) The reparameterization invariance is automatically satisfied since, by construction, the coefficient of the kinetic term is supposed to be constant.
(ii) Equation (5), with ${\rm v}_2 = {\rm v}_3 = 0$, does not depend on any free parameter like b.

Consequently, there is no ambiguity: to any value of N, corresponds a unique value of ν and ω while $\eta = 0$. As already mentioned in [42], it is noteworthy that the values I obtain (see also [41]) for the critical and subcritical exponents (without optimization since there are no free parameters at hand) agree, 'to all published digits', with the 'optimized' values obtained in [43] from a study of an exact RG equation of the second family. Hence, for the numerical results I have obtained at order LPA for ν, ω and other subleading critical exponents ω_n, the reader is referred to table 1 of [43].

At order ∂^2, once the arbitrariness of Z_0 has been removed via the definition of $\eta^{\rm opt}$, the values of the critical exponents still depend on b in such a way that, for a given value of N, it is impossible to define 'preferred' values. For example η (from now on, η stands for $\eta^{\rm opt}$) depends on b almost linearly.

Due to the spreading out over several orders of the effects of b, the study of the convergence of the derivative expansion relies rather on the consideration of several orders. Now the

Table 1. Critical exponent estimates (for $d = 3$) from the O(N) Wilson–Polchinski exact RG equation expanded up to $O(\partial^2)$ in the derivative expansion for two values of the free parameter b.

	$b = 0.03$			$b = 0.11$		
N	η	ν	ω	η	ν	ω
1	0.010 06	0.6223	0.7755	0.024 94	0.5994	0.8740
2	0.008 66	0.6723	0.7266	0.022 63	0.6388	0.7656
3	0.007 21	0.7238	0.7132	0.020 07	0.6838	0.7156
4	0.005 92	0.7713	0.7223	0.017 39	0.7313	0.7010
5	0.004 86	0.8111	0.7437	0.014 89	0.7762	0.7088
6	0.004 04	0.8424	0.7695	0.012 75	0.8138	0.7307
7	0.003 42	0.866	0.7946	0.011 00	0.8440	0.7576
8	0.002 94	0.8849	0.8172	0.009 60	0.8673	0.7842
9	0.002 56	0.8993	0.8368	0.008 48	0.8854	0.8081
10	0.002 27	0.9107	0.8533	0.007 57	0.8995	0.8292
11	0.002 04	0.9214	0.8711	0.006 83	0.9108	0.8463
12	0.001 84	0.9293	0.8836	0.006 21	0.9198	0.8613
15	0.001 42	0.9438	0.9060	0.004 88	0.9387	0.8942
20	0.001 02	0.9596	0.9330	0.003 58	0.9563	0.9258

Table 2. Same as table 1 (two other values of b).

	$b = 0.170\,11$			$b = 0.256\,11$		
N	η	ν	ω	η	ν	ω
1	0.035 53	0.5850	0.9515	0.049 79	0.5677	1.0906
2	0.032 78	0.6176	0.7917	0.046 70	0.5908	0.8273
3	0.029 80	0.6574	0.7115	0.043 71	0.6242	0.7288
4	0.026 43	0.7033	0.6806	0.039 99	0.6664	0.6570
5	0.023 00	0.7506	0.6809	0.035 65	0.7175	0.6538
6	0.019 86	0.7939	0.7040	0.031 75	0.7737	0.6682
7	0.017 19	0.8291	0.7331	0.027 72	0.8224	0.7137
8	0.015 02	0.8565	0.7641	0.023 52	0.8565	0.7456
9	0.013 27	0.8779	0.7938	0.021 50	0.8928	0.8037
10	0.011 85	0.8944	0.8194	0.019 37	0.9132	0.8165
11	0.010 68	0.9075	0.8413	0.017 34	0.9357	0.8505
12	0.009 72	0.9180	0.8596	0.015 88	0.9552	0.8741
15	0.007 79	0.9552	0.9209	—	—	—

order ∂^2 is still too low to allow an appreciation of the convergence of the derivative expansion. Instead of presently producing one 'best' estimate for each critical exponent at a given value of N, it is preferable to maintain the freedom of b in order to better emphasize the early beginnings of some criteria of convergence if any (see section 3.3).

Tables 1 and 2 display the estimates of η, ν and ω as obtained for four values of b and N varying from 1 to 20 (while $d = 3$).

For $N = 1$, the results of [13] (preferred value of η^{opt} for $b = 0.11$) are obtained again. In this latter work I had proposed a criterion of choice of the value of b which gave a preferred value of η. It was based on the idea of Golner [15] of a global minimization of the magnitude of the function $Z(\phi)$ (i.e., $\mathrm{v}_2(\phi)$). However, the extension of this criterion to $N > 1$ is not easy to implement because the function $Z(\phi)$ has been split into two parts ($Z(\rho)$ and $Y(\rho)$).

The further discussion of these results is left to section 3.3.

Table 3. Compared to LPA on the left-hand side of the arrows, the order ∂^2 (right-hand side of the arrows) induces a splitting into two of the subcritical exponents of degree higher than ω (values for $d = 3$, $N = 1$ and $b = 0.11$). The LPA values may be found in [43].

ω	ω_2	ω_3	ω_4
$0.6557 \to 0.8740$	$3.18 \to \begin{cases} 2.83 \\ 3.62 \end{cases}$	$5.91 \to \begin{cases} 5.70 \\ 6.40 \end{cases}$	$8.80 \to \begin{cases} 7.30 \\ 9.6 \end{cases}$

Table 3 displays a comparison between the LPA and $\mathrm{O}(\partial^2)$ results for $d = 3$, $N = 1$ and $b = 0.11$ of the series of subcritical exponents ω_n (with $\omega_1 = \omega$). Compared to LPA, the order $\mathrm{O}(\partial^2)$ has increased the potential number of subcritical exponents (due to the supplementary terms in the truncated action considered). For instance, if one adopts the dimensional analysis of perturbation theory, then at order LPA a term like ϕ^6 contributing to $S[\phi]$ is associated with the subcritical ω_2, while a ϕ^8-term generates ω_3. But at order $\mathrm{O}(\partial^2)$, new terms involving two derivatives contribute to $S[\phi]$, and a term proportional to $\phi^2(\partial\phi)^2$ induces *a priori* an in-between correction exponent. Despite important differences between the classical dimensions of the couples ϕ^6 and $\phi^2(\partial\phi)^2$, table 3 clearly shows that, for $N = 1$, the order ∂^2 induces a splitting of the LPA values of the subcritical exponents ω_n for $n > 1$ (ω is, in the perturbative approach, associated to the unique ϕ^4 coupling and is not, fortunately for the perturbative framework, subject to this splitting). This simple splitting is presumably not preserved for other values of N due to the supplementary contribution of $Y(\phi)$.

3.3. Comparison with other studies, discussions and conclusion

Figure 1 shows the evolution of η with N from different works. The results obtained from the resummation of six [29] and seven [30] orders of the perturbation field theory serves the purpose of standards (other accurate estimates of the critical exponents, especially for $N \leqslant 4$, exist in the literature; for a review see [5]). One sees that the present study yields generally small values of η compared to the standards except for small N and for the highest values of b. The evolution of η with N is smoother than in the work of Morris and Turner [26] but, as in this latter work, the non-monotonic behaviour of the standards (responsible for the maximum of η about $N = 2$ or 3) is not reproduced. Instead, the results of Gersdorff and Wetterich [27] are better. The present results are, however, not so bad if one keeps in mind that the first estimate of η in the derivative expansion is given at order ∂^2. In particular, figure 1 also shows recent estimates from the resummation of three orders of the perturbative series using an efficient method [44]. One sees that the present estimates withstand the comparison (except the monotonic evolution with N). Notice also, for N fixed, the monotonic evolution of η with b already mentioned.

The results for the critical exponent ν are more interesting to discuss because the order ∂^2 provides its second estimate. Figure 2 shows the evolution with N of ν at order ∂^2 compared to the results at order LPA (obtained in the present work); the standards [29, 30] are reproduced also. Again, one observes that the results for large N are not as good as for small values. However, for these latter values, one clearly sees that there is a range of values of b where the two estimates at orders LPA and ∂^2 flank the standards, and another range where the two present estimates are on the same side (with respect to the standards). This is a phenomenon often observed in convergent series, the elements of which depend on a free parameter (like b), but the resumed series does not: on varying the free parameter, one may observe monotonic or alternate approaches to the limit. These features may be used to determine error bars.

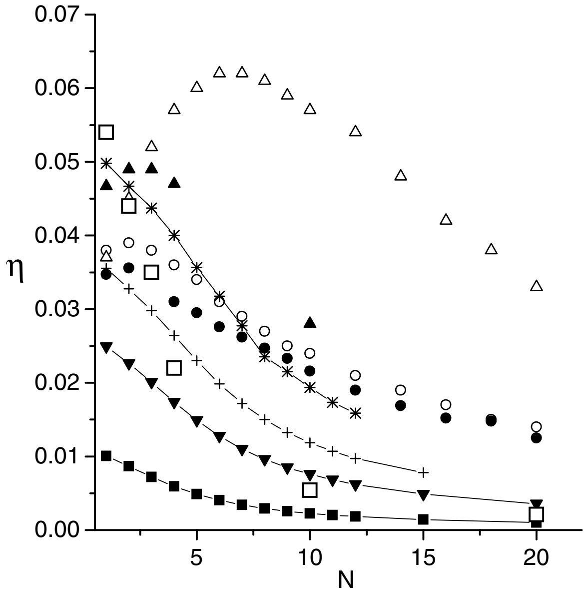

Figure 1. Critical exponent η as function of N ($d = 3$). Open and full circles represent the standards; they were obtained from the resummation of six (○ from [29]) or seven (● from [30]) orders of the perturbation field theory. Recent estimates of [44] from the same perturbative series at order three only are also represented (△). The other estimates are from the exact RG equation expanded up to $O(\partial^2)$ in the derivative expansion: full triangle up from [27], □ from [26]. The points linked by straight lines are from the present work: ■ for $b = 0.03$, full triangle down for $b = 0.11$, + for $b = 0.170\,11$, $*$ for $b = 0.256\,11$. See text for discussion.

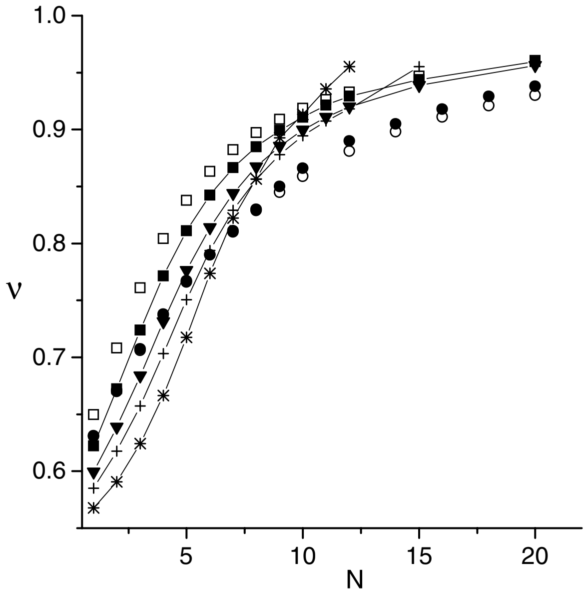

Figure 2. Critical exponent ν as function of N ($d = 3$). Open and full circles represent the standards as in figure 1. The other estimates are from the present work: □ from order LPA; the points linked by straight lines are from $O(\partial^2)$ in the derivative expansion: ■ for $b = 0.03$, full triangle down for $b = 0.11$, + for $b = 0.170\,11$, $*$ for $b = 0.256\,11$. See text for discussion.

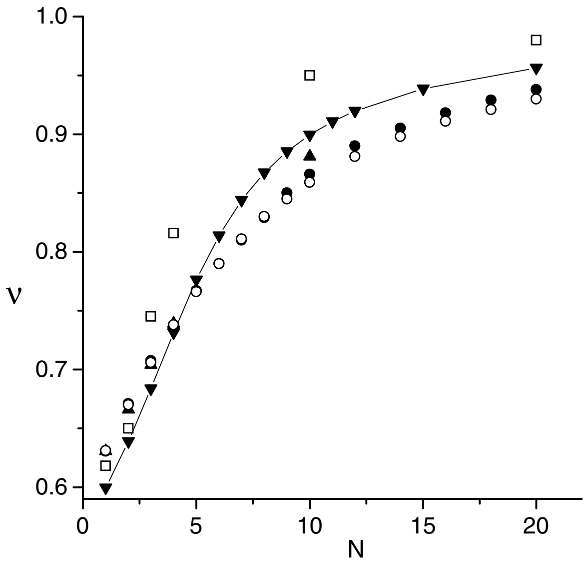

Figure 3. Critical exponent ν as function of N ($d = 3$). Open and full circles represent the standards as in figure 1. The other estimates are from the exact RG equation expanded up to O(∂^2) in the derivative expansion: full triangle up from [27], □ from [26]. The estimates of the present study for $b = 0.11$ are reported (full triangles down linked by straight lines; see figure 2 for the other values of b). See text for discussion.

At present, one additional order would be necessary to propose such error bars. Figure 2 shows also that, when N increases, the dependence of ν on b becomes non-monotonic. This is interesting since such extrema may indicate a vestige of the primarily independence on b of the exact RG equation. Why this effect does not occur at small values of N is not explained. Once more, at least one supplementary order of the derivative expansion would be necessary to understand this point.

Figure 3 shows the results for ν coming from [26] and [27] compared to the present results for $b = 0.11$ and the standards [29, 30]. One observes that the present results are globally better than in [26], and that again the estimates of Gersdorff and Wetterich [27] are excellent (for small N the points almost coincide with the standards).

Figure 4 shows the present results for ω at order LPA and ∂^2 compared to the standards [28, 30]. This figure is the matching piece to figure 2 and the same kinds of remarks stand: monotonic and alternate approaches to the standards at fixed N exist as well as non-monotonic dependences on b. The magnitude of these effects is larger than for ν and the accuracy is worse, but this is expected for a subleading eigenvalue: the accuracy decreases as the order of the eigenvalue increases.

Figure 5 shows the results for ω coming from [26] compared to the present results for $b = 0.11$ and the standards [28, 30]. One observes that the present results are much better than in [26].

One may regret that Gersdorff and Wetterich [27], who obtained excellent values for ν and η, had not estimated ω. As already said, this study [27] does not follow the conventional approach defined in the introduction. In particular, nothing is said on the way the reparameterization invariance is accounted for. In fact, instead of leaving free the value of $Z(0) = Z_0$ to get a function $\eta(Z_0)$, the procedure followed in [27] was to attach the determination of η to the minimum of the potential. This condition fixes Z_0 and the

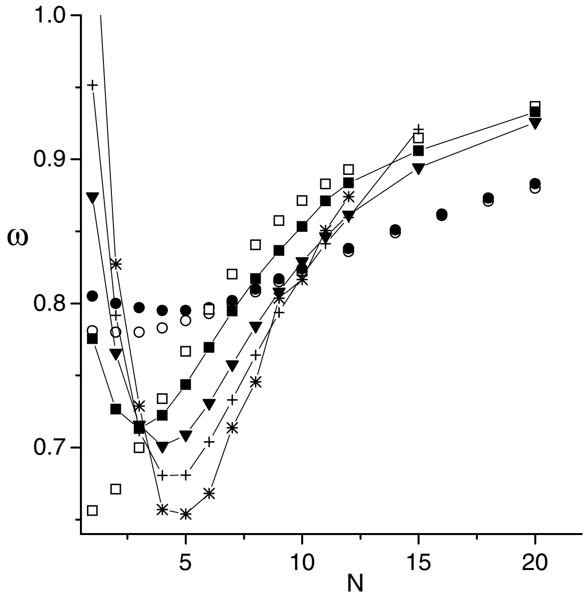

Figure 4. Subcritical exponent ω as a function of N ($d = 3$). Open and full circles represent the standards as in figure 1 but the open circles (○) are from [28]. The other estimates are from the present work: □ from order LPA; the points linked by straight lines are from $\mathrm{O}(\partial^2)$ in the derivative expansion: ■ for $b = 0.03$, full triangle down for $b = 0.11$, + for $b = 0.170\,11$, $*$ for $b = 0.256\,11$. See text for discussion.

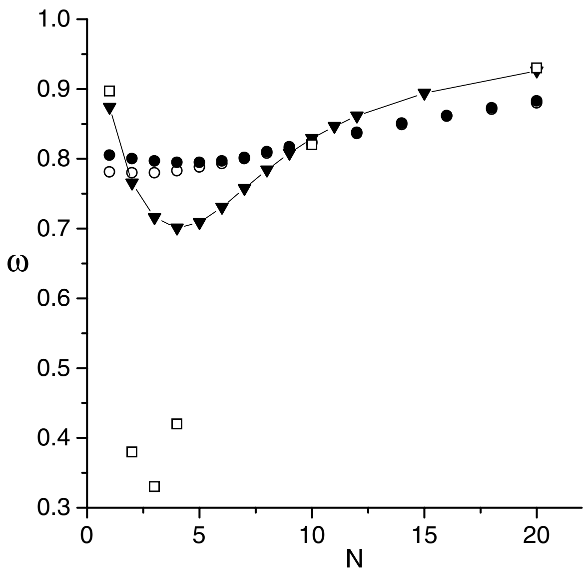

Figure 5. Subcritical exponent ω as function of N ($d = 3$). Open and full circles represent the standards as in figure 4. The other estimates are from the exact RG equation expanded up to $\mathrm{O}(\partial^2)$ in the derivative expansion: □ from [26]. The estimates of the present study for $b = 0.11$ are reported (full triangles down linked by straight lines; see figure 4 for the other values of b). See text for discussion.

arbitrariness carried by the reparameterization invariance is implicitly removed this way. Because Morris and Turner [26] have considered an equation of the same family and have obtained disappointing results, I think that this particular way of choosing η^{opt} could be the main reason for the excellent estimates of the critical exponents obtained in [27]. It will be interesting to adapt it to the study of the Wilson–Polchinski RG equation.

To conclude, the derivative expansion at order $\mathrm{O}(\partial^2)$ already displays a tendency to converge. This must be confirmed by considering the next order, which is in progress [32]. The study of [25] which for $N = 1$ follows the procedure of [27] and the optimization process of [38] is very encouraging. I think that the Wilson–Polchinski RG equation, which is the simplest exact RG equation, is better adapted to the estimation of the critical exponents. Further studies should be undertaken with a view to better determining the status of the derivative expansion.

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